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Discretized Normal Approximation for the Number of Descents

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Abstract

This paper explores the probability approximation of the number of descents in a random permutation. It is known that the distribution of the number of descents can be approximated by a normal distribution under Kolmogorov distance. Subsequently, an explicit constant is provided in the bound of 13.42. The objective is to prove a similar result but using a stronger distance, namely, the total variation distance. The Stein's method and the exchangeable pair transformation has been used to give a bound for discretized normal approximation for the distribution of number of descents under the total variation distance. The result obtained gave an improved constant of 10.71.

Keywords: the number of descents; normal approximation; discretized normal approximation; Kolmogorov distance; total variation distance; Stein's method; exchangeable pair.

1 Introduction

Let S_n be the set of permutations on $\{1,2,\dots,n\}$. A uniform random permutation π is a uniform random vector whose value is in S_n, i.e.,

$$P\big(\pi=(i_1,i_2,\dots,i_n)\big)=\frac{1}{n!},\qquad \text{for all}\qquad (i_1,i_2,\dots,i_n)\in S_n.$$

Then $\pi=\big(\pi(1),\pi(2),\dots,\pi(n)\big)$ is a random vector such that $P(\pi(i)=j)=\frac{1}{n}$ for $j\in\{1,2,\dots,n\}$ and $\pi(i)\neq\pi(j)$ for $i\neq j$. In this work, we are interested in the number of descents of π which is defined by,

$$D_n(\pi)=\text{the number of pairs }\ (i,i+1)\quad\text{with}\quad 1\le i\le n-1\quad\text{and}\quad \pi(i)>\pi(i+1).$$

In statistics, an interesting application is the number of descents of shuffling cards [4]. In addition, in biology, the number of descents of the DNA sequence has been studied [13].

For any non-empty subset A of $\{0,1,\dots,n-1\}$, it is difficult to find $P\big(D_n(\pi)\in A\big)$ directly. We only know that,

$$E(D_n(\pi))=\frac{n-1}{2},\quad\text{and}\quad Var(D_n(\pi))=\frac{n+1}{12},\tag{1}$$

([9], pp. 71). Therefore, accurate approximation of the distribution of $D_n(\pi)$ is desirable. In 2004, Fulman [9] approximated $P\left(\frac{D_n(\pi)-E(D_n(\pi))}{\sqrt{Var(D_n(\pi))}}\le z\right)$ for $z\in\mathbb{R}$ by the standard normal random variable Z. He used Stein's method and the exchangeable pair technique to show that,

$$\sup_{z\in\mathbb{R}}\left|P\left(\frac{D_n(\pi)-E(D_n(\pi))}{\sqrt{Var(D_n(\pi))}}\le z\right)-P(Z\le z)\right|\le\frac{C}{\sqrt{n}},\tag{2}$$

where C is a positive constant. In 2015, Chuntee and Neammanee [3] followed the argument of Fulman [9] to show that C in (2) is 1096 and used the technique of Neammanee and Rattanawong [15] reduce the constant to 13.42. Since then, this work has been improved by many mathematicians (see [16]).

Since the values of $D_n(\pi)$ are in the set $\{0,1,\dots,n-1\}$, it is sufficient to approximate $P(D_n(\pi)\in A)$, where $A\subseteq\{0,1,\dots,n-1\}$. If A is of the form $A_k=\{0,1,\dots,k\}$ for some $k\in\mathbb{N}\cup\{0\}$, then $P(D_n(\pi)\in A_k)=P\left(\frac{D_n(\pi)-E(D_n(\pi))}{\sqrt{Var(D_n(\pi))}}\le\frac{k-E(D_n(\pi))}{\sqrt{Var(D_n(\pi))}}\right)$ and we can apply (2). But in this study, we need to approximate $P(D_n(\pi)\in A)$ for an arbitrary subset A of $\{0,1,\dots,n-1\}$ by the distribution of a suitable random variable U, i.e.,

$$\sup_{A\subseteq\{0,1,\dots,n-1\}}\left|P\left(D_n(\pi)\in A\right)-P(U\in A)\right|.\tag{3}$$

Note that the distance in (2) is called the Kolmogorov distance between the distribution of $\frac{D_n(\pi)-E(D_n(\pi))}{\sqrt{Var(D_n(\pi))}}$ and Z and (3) is called the total variation distance between the distribution of $D_n(\pi)$ and U.

One can verify that if $A = \{0, 1, \ldots, n-1\}$, then $P(Z_{\mu,\sigma^2} \in A) = 0$ and $P(D_n(\pi) \in A) = 1$, where $Z_{\mu,\sigma^2} \sim N(\mu, \sigma^2)$ is a normal random variable with mean μ and variance σ^2. This implies that,

$$\sup_{A \subseteq \{0,1,\ldots,n-1\}} \left| P\left(D_n(\pi) \in A\right) - P(Z_{\mu,\sigma^2} \in A) \right| = 1,$$

which does not converge to 0. This shows that a normal distribution is not suitable to approximate the distribution of $D_n(\pi)$ with total variation distance and we need to find an other distribution. In general, there is not only a normal distribution but also other types of distributions, for example, the Rayleigh distribution [5], log-normal distribution [14], translated Poisson distribution [19, 20], and binomial distribution [22]. In this work we will use the discretized normal distribution as our approximating distribution. The discretized normal distribution N^d_{μ,σ^2} with parameters μ and σ^2 has probability mass function,

$$P(N^d_{\mu,\sigma^2} = k) = P\left(k - \frac{1}{2} \leq Z_{\mu,\sigma^2} < k + \frac{1}{2}\right), \quad k \in \mathbb{Z}.$$

We will use the exchangeable pair and Stein's method to obtain the total variation distance between the distribution of $D_n(\pi)$ and a discretized normal distribution, a concept introduced by Fang [8]. Theorem 1.1 stated below is

Theorem 1.1. *Let N^d_{μ,σ^2} be the discretized normal distribution with $\mu = E(D_n(\pi))$ and $\sigma^2 = Var(D_n(\pi))$. Then,*

$$\sup_{A \subseteq \{0,1,\ldots,n-1\}} \left| P\left(D_n(\pi) \in A\right) - P(N^d_{\mu,\sigma^2} \in A) \right| \leq \frac{10.71}{\sqrt{n}}.$$

Remark 1.1. *In the case of $A = A_k$, by (2) and Theorem 1.1, $P(D_n(\pi) \in A)$ can be approximated by the standard normal and the discretized normal distribution. However, the constant in the discretized normal approximation is sharper.*

The rest of this paper is structured as follows: Section 2 introduces Stein's method, the exchangeable pair approach, and some properties of descent. Section 3 provides the proof of our main result.

2 Auxiliary Results

2.1 Stein's method

The technique used to establish Theorem 1.1 is Stein's method. It is employed for approximating the distribution of a random variable using an appropriate distribution. This method was introduced by Stein [23] in 1972 when the limit distribution is normal. Stein's method is well-known and powerful in normal approximation and has been extended in several directions. Chen [1] developed Stein's method for the Poisson approximation in 1975. After that, many mathematicians have improved this method for various distributions, including the half-normal distribution [7], gamma distribution [11], and Laplace distribution [17].

Let Z_{μ,σ^2} be a normal random variable with mean μ and positive variance σ^2 and let $\mathbb{F}$ be the set of continuous and piecewise differentiable function $f : \mathbb{R} \to \mathbb{R}$ with $E|f'(Z_{\mu,\sigma^2})| < \infty$. The method relies on the following differential equation called Stein's equation, i.e.,

$$\sigma^2 f'(w) - (w - \mu) f(w) = h(w) - Eh(Z_{\mu,\sigma^2}), \tag{4}$$

where h is a is a real valued measurable function with $E|h(Z_{\mu,\sigma^2})| < \infty$ and $f \in \mathbb{F}$.

From (4), we see that,

$$\left(\sigma^2 e^{-\frac{(w-\mu)^2}{2\sigma^2}} f(w)\right)' = e^{-\frac{(w-\mu)^2}{2\sigma^2}} \Big[h(w) - Eh(Z_{\mu,\sigma^2})\Big].$$

This implies that, the bounded solution f_h of (4) is given by,

$$f_h(w) = \frac{1}{\sigma^2} e^{\frac{(w-\mu)^2}{2\sigma^2}} \int_{-\infty}^{w} \Big[h(x) - Eh(Z_{\mu,\sigma^2})\Big] e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx. \tag{5}$$

In the case where $\mu = 0$, $\sigma^2 = 1$ and h is bounded, Stein ([24], pp. 25) showed that,

$$||f_h|| \leq \sqrt{\frac{\pi}{2}} ||h - Eh(Z_{0,1})|| \quad \text{and} \quad ||f_h'|| \leq 2||h - Eh(Z_{0,1})||,$$

where $||f|| = \sup_{x\in\mathbb{R}} |f(x)|$. We also note that in case that $f_h'(w)$ does not exist, we will define $f_h'(w)$ by (4), i.e.,

$$f_h'(w) = \frac{1}{\sigma^2} \Big[h(w) - Eh(Z_{\mu,\sigma^2}) + (w-\mu) f_h(w)\Big],$$

(see[24] pp. 26 for more details). For arbitrary μ and σ^2, we may use the the technique from Stein [24] to prove the following lemma;

Lemma 2.1. *For a real valued measurable function h, let f_h be the bounded solution of the Stein equation (4). If h is bounded, then,*

1. $||f_h|| \leq \sqrt{\frac{\pi}{2}} \cdot \frac{1}{\sigma} ||h - Eh(Z_{\mu,\sigma^2})||$.
2. $||f_h'|| \leq \frac{2}{\sigma^2} ||h - Eh(Z_{\mu,\sigma^2})||$.

Proof. We can follow the argument of Stein ([24], pp. 25–26). □

For a measurable subset A of $\mathbb{R}$, if we choose $h = h_A$ where $h_A : \mathbb{R} \to \mathbb{R}$ defined by,

$$h_A(w) = \begin{cases} 1, & \text{if} \quad w \in A, \\ 0, & \text{if} \quad w \notin A, \end{cases}$$

from (4), we have

$$\sigma^2 f'(w) - (w-\mu) f(w) = h_A(w) - Eh_A(Z_{\mu,\sigma^2}). \tag{6}$$

Replacing w by a random variable W in (6) and taking expectations, (6) becomes,

$$\sigma^2 E f_{h_A}'(W) - E(W-\mu) f_{h_A}(W) = P(W \in A) - P(Z_{\mu,\sigma^2} \in A), \tag{7}$$

where f_{h_A} is the solution of (6). According to (5),we have

$$f_{h_A}(w) = \frac{1}{\sigma^2} e^{\frac{(w-\mu)^2}{2\sigma^2}} \int_{-\infty}^{w} [h_A(x) - Eh_A(Z_{\mu,\sigma^2})] e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx. \tag{8}$$

Futhermore, from Lemma 2.1 and the fact that $0 \le h_A \le 1$, we have

$$||f_{h_A}|| \le \sqrt{\frac{\pi}{2}} \cdot \frac{1}{\sigma} \quad \text{and} \quad ||f'_{h_A}|| \le \frac{2}{\sigma^2}. \tag{9}$$

From (7), we can bound $|\sigma^2 E f'_{h_A}(W) - E(W-\mu) f_{h_A}(W)|$ instead $|P(W \in A) - P(Z_{\mu,\sigma^2} \in A)|$. This technique is called Stein's method.

For any subset A of $\{0, 1, \dots, n-1\}$, we let $\bar{A} = \bigcup_{z \in A} \left[z - \frac{1}{2}, z + \frac{1}{2}\right)$. Since $D_n(\pi)$ and N^d_{μ,σ^2} are integer-valued random variables, we obtain

$$P(D_n(\pi) \in A) = P(D_n(\pi) \in \bar{A}),$$

and

$$P(N^d_{\mu,\sigma^2} \in A) = \sum_{z \in A} P(N^d_{\mu,\sigma^2} = z) = \sum_{z \in A} P\left(z - \frac{1}{2} \le Z_{\mu,\sigma^2} < z + \frac{1}{2}\right) = P(Z_{\mu,\sigma^2} \in \bar{A}).$$

These facts imply that,

$$P(D_n(\pi) \in A) - P(N^d_{\mu,\sigma^2} \in A) = P(D_n(\pi) \in \bar{A}) - P(Z_{\mu,\sigma^2} \in \bar{A}). \tag{10}$$

Hence, from (7) and (10), we get

$$E\sigma^2 f'_{h_{\bar{A}}}(D_n(\pi)) - E(D_n(\pi) - \mu) f_{h_{\bar{A}}}(D_n(\pi)) = P(D_n(\pi) \in A) - P(N^d_{\mu,\sigma^2} \in A), \tag{11}$$

where $f_{h_{\bar{A}}}$ is defined by (8) when $A = \bar{A}$.

2.2 Exchangeable pair

In Stein's method, there are several approaches to justify the approximation of the distribution of a random variable, for instance, zero bias and size bias approaches [12, 2], as well as exchangeable pair [21, 24].

In this work, we consider the exchangeable pair introduced by Diaconis [6] and Stein [24]. This approach has been used in many different ways such as Jack measure [10], and Markov structure [18]. The exchangeable pair refers to a pair of random variables that shows a symmetry property, namely, (W, W') is an exchangeable pair if,

$$(W, W') \stackrel{d}{=} (W', W),$$

where $\stackrel{d}{=}$ signifies equality in distribution.

Note that if a pair (W, W') is an exchangeable pair, then W and W' are identically distributed but the converse is not true. Chen et al. ([2], pp. 21) gave the following property of exchangeable pairs.

Proposition 2.1. *Let (W, W') be an exchangeable pair. If $g : \mathbb{R}^2 \to \mathbb{R}$ is an antisymmetric function, i.e., $g(x, y) = -g(y, x)$ for any $x, y \in \mathbb{R}$, then,*

$$Eg(W, W') = 0.$$

As in Chen et al. [2] and Fang [8], the exchangeable pair (W, W') is called a $\lambda-$Stein pair for $\lambda \in \mathbb{R}^+$, if there exits a random variable R such that,

$$E^W(W - W') = \lambda(W - \mu) + \sigma E^W(R),$$

for $\mu = E(W), \sigma^2 = Var(W)$. Fang ([8], pp. 1411) gave a property of $\lambda-$Stein pair as follows:

Proposition 2.2. *Let (W, W') be a $\lambda-$Stein pair. Then,*

$$\frac{1}{2\lambda}E(W' - W)\big(f(W') - f(W)\big) - \frac{\sigma}{\lambda}Ef(W)R = E(W - \mu)f(W),$$

for all $f : \mathbb{R} \to \mathbb{R}$ such that the expectations exist.

To prove our main result, we need to construct an exchangeable pair of $D_n(\pi)$.

Let,

$$U_n(\pi) = \frac{D_n(\pi) - E(D_n(\pi))}{\sqrt{Var(D_n(\pi))}}. \tag{12}$$

Fulman [9] constructs an exchangeable pair of $U_n(\pi)$ as following. Let I be a uniform random variable and π be a random permutation on $\{1, 2, \ldots, n\}$. Define a random permutation π' as follows:

$$\pi'(i) = \begin{cases} \pi(i), & \text{if} \quad i \notin \{I, I+1, \ldots, n\}, \\ \pi(i+1), & \text{if} \quad i \in \{I, I+1, \ldots, n-1\}, \\ \pi(I), & \text{if} \quad i = n. \end{cases}$$

Then,

$$\big(U_n(\pi'), U_n(\pi)\big) \text{ is an exchangeable pair.} \tag{13}$$

Moreover, Fulman [9] and Chuntee and Neammanee [3] gave some properties of $U_n(\pi)$ as the following lemma;

Lemma 2.2.

1. $|U_n(\pi') - U_n(\pi)| \leq \dfrac{2\sqrt{3}}{\sqrt{n+1}}$.
2. $E^\pi(U_n(\pi') - U_n(\pi)) = -\dfrac{2}{n}U_n(\pi)$.
3. $E(U_n(\pi') - U_n(\pi))^2 = \dfrac{4}{n}$.
4. $E[E^\pi(U_n(\pi') - U_n(\pi))^2]^2 = \dfrac{16}{n^2}\left[1 + \dfrac{8}{5(n+1)}\right]$.

Proof. See page 68–72 in [9] and page 2314–2315 in [3]. □

By (12), we have

$$D_n(\pi) = \sqrt{Var(D_n(\pi))}U_n(\pi) + E(D_n(\pi)). \tag{14}$$

Note that, if a pair (W, W') is an exchangeable pair, then $(aW+b, aW'+b), a, b \in \mathbb{R}$ is an exchangeable pair. Hence, by (13), we get

$$\left(\sqrt{Var(D_n(\pi))}U_n(\pi') + E(D_n(\pi)), \sqrt{Var(D_n(\pi))}U_n(\pi) + E(D_n(\pi))\right), \tag{15}$$

is an exchangeable pair. Let,

$$D_n(\pi') = \sqrt{Var(D_n(\pi))}U_n(\pi') + E(D_n(\pi)). \tag{16}$$

Then, by (14), (15) and (16),

$$\big(D_n(\pi'), D_n(\pi)\big) \text{ is an exchangeable pair.}$$

To prove our main theorem, we give some properties of $D_n(\pi)$ in the following lemma;

Lemma 2.3. *Let $\mu = E(D_n(\pi))$ and $\sigma^2 = Var(D_n(\pi))$. Then,*

1. $|D_n(\pi') - D_n(\pi)| \leq 1$.
2. $E^{\pi}\big(D_n(\pi') - D_n(\pi)\big) = -\frac{2}{n}\big(D_n(\pi) - \mu\big)$.
3. $E\big(D_n(\pi') - D_n(\pi)\big)^2 = \frac{4\sigma^2}{n}$.
4. $E\Big[E^{\pi}\big(D_n(\pi') - D_n(\pi)\big)^2\Big]^2 = \frac{16\sigma^4}{n^2}\left[1 + \frac{8}{5(n+1)}\right]$.

Proof. To prove the lemma, we note from (1), (14) and (16) that,

$$\begin{aligned} |D_n(\pi') - D_n(\pi)| &= \left|\left(\sqrt{Var(D_n(\pi))}U_n(\pi') + E(D_n(\pi))\right) - \left(\sqrt{Var(D_n(\pi))}U_n(\pi) + E(D_n(\pi))\right)\right|, \\ &= \sqrt{Var(D_n(\pi))}|U_n(\pi') - U_n(\pi)|, \\ &= \sqrt{\frac{n+1}{12}}|U_n(\pi') - U_n(\pi)|. \end{aligned} \tag{17}$$

Then,

1. follows from (17) and Lemma 2.2(1).
2. follows from (17) and Lemma 2.2(2).
3. follows from (17) and Lemma 2.2(3).
4. follows from (17) and Lemma 2.2(4).

□

3 Proof of Main Result

In this section, we prove our main theorem by using Stein's method, exchangeable pair and some techniques from Fang [8]. From now on, let $h = h_{\bar{A}}$ and $f = f_{h_{\bar{A}}}$ where $\mu = E(D_n(\pi))$ and $\sigma^2 = Var(D_n(\pi))$.

Proof. By Lemma 2.3(2), it follows that $\big(D_n(\pi'), D_n(\pi)\big)$ is a $\lambda-$Stein pair with $\lambda = \dfrac{2}{n}$ and $R = 0$. From Proposition 2.2, we obtain

$$\begin{aligned}
&E\big(D_n(\pi) - \mu\big) f(D_n(\pi)) \\
&= \frac{n}{4} E\big(D_n(\pi') - D_n(\pi)\big)\Big[f(D_n(\pi')) - f(D_n(\pi))\Big], \\
&= \frac{n}{4} E\big(D_n(\pi') - D_n(\pi)\big) \int_0^{D_n(\pi') - D_n(\pi)} \big[f'(D_n(\pi) + t)\big] dt, \\
&= \frac{n}{4} E\big(D_n(\pi') - D_n(\pi)\big) \int_0^{D_n(\pi') - D_n(\pi)} \big[f'(D_n(\pi) + t)\big] dt \\
&\quad + \frac{n}{4} E\big(D_n(\pi') - D_n(\pi)\big)^2 f'(D_n(\pi)) - \frac{n}{4} E\big(D_n(\pi') - D_n(\pi)\big) \int_0^{D_n(\pi') - D_n(\pi)} f'(D_n(\pi)) dt, \\
&= \frac{n}{4} E\big(D_n(\pi') - D_n(\pi)\big)^2 f'(D_n(\pi)) \\
&\quad + \frac{n}{4} E\big(D_n(\pi') - D_n(\pi)\big) \int_0^{D_n(\pi') - D_n(\pi)} \Big[f'(D_n(\pi) + t) - f'(D_n(\pi))\Big] dt. \qquad (18)
\end{aligned}$$

Using (4), we have

$$\begin{aligned}
&f'\big(D_n(\pi) + t\big) - f'(D_n(\pi)) \\
&= \frac{1}{\sigma^2}\Big[\big(D_n(\pi) + t - \mu\big) f(D_n(\pi) + t) + h(D_n(\pi) + t) - Eh(Z_{\mu,\sigma^2})\Big] \\
&\quad - \frac{1}{\sigma^2}\Big[\big(D_n(\pi) - \mu\big) f(D_n(\pi)) + h(D_n(\pi)) - Eh(Z_{\mu,\sigma^2})\Big], \\
&= \frac{1}{\sigma^2}\Big[\big(D_n(\pi) + t - \mu\big) f(D_n(\pi) + t) - (D_n(\pi) - \mu) f(D_n(\pi)) + h(D_n(\pi) + t) - h(D_n(\pi))\Big].
\end{aligned}$$

From this fact and (18), we have

$$\begin{aligned}
&E\sigma^2 f'(D_n(\pi)) - E(D_n(\pi) - \mu) f(D_n(\pi)) \\
&= \sigma^2 E f'(D_n(\pi)) - \frac{n}{4} E(D_n(\pi') - D_n(\pi))^2 f'(D_n(\pi)) \\
&\quad - \frac{n}{4\sigma^2} E\big(D_n(\pi') - D_n(\pi)\big) \int_0^{D_n(\pi') - D_n(\pi)} \Big[(D_n(\pi) + t - \mu) f(D_n(\pi) + t) \\
&\quad - (D_n(\pi) - \mu) f(D_n(\pi)) + h(D_n(\pi) + t) - h(D_n(\pi))\Big] dt, \\
&= E f'(D_n(\pi)) \Big(\sigma^2 - \frac{n}{4}(D_n(\pi') - D_n(\pi))^2\Big) \\
&\quad - \frac{n}{4\sigma^2} E\big(D_n(\pi') - D_n(\pi)\big) \int_0^{D_n(\pi') - D_n(\pi)} \Big[(D_n(\pi) + t - \mu) f(D_n(\pi) + t) \\
&\quad - (D_n(\pi) - \mu) f(D_n(\pi)) + h(D_n(\pi) + t) - h(D_n(\pi))\Big] dt, \\
&= \triangle_1 - \triangle_2 - \triangle_3 - \triangle_4, \qquad (19)
\end{aligned}$$

where,

$$\triangle_1 = Ef'(D_n(\pi))\left(\sigma^2 - \frac{n}{4}\left(D_n(\pi') - D_n(\pi)\right)^2\right), \tag{20}$$

$$\triangle_2 = \frac{n}{4\sigma^2}E(D_n(\pi') - D_n(\pi))\int_0^{D_n(\pi')-D_n(\pi)} tf(D_n(\pi)+t)dt, \tag{21}$$

$$\triangle_3 = \frac{n}{4\sigma^2}E(D_n(\pi') - D_n(\pi))\int_0^{D_n(\pi')-D_n(\pi)} \Big[D_n(\pi) - \mu\Big]\Big[f(D_n(\pi)+t) - f(D_n(\pi))\Big]dt, \tag{22}$$

$$\triangle_4 = \frac{n}{4\sigma^2}E(D_n(\pi') - D_n(\pi))\int_0^{D_n(\pi')-D_n(\pi)} \Big[h(D_n(\pi)+t) - h(D_n(\pi))\Big]dt. \tag{23}$$

We give a bound for each error as follows:
Bounding (20): By Lemma 2.3(3) and Lemma 2.3(4), we have

$$\begin{aligned}
&E\left[\sigma^2 - \frac{n}{4}E^\pi\left(D_n(\pi') - D_n(\pi)\right)^2\right]^2 \\
&= \sigma^4 - \frac{n\sigma^2}{2}E\big(D_n(\pi') - D_n(\pi)\big)^2 + \frac{n^2}{16}E\Big[E^\pi\big(D_n(\pi') - D_n(\pi)\big)^2\Big]^2, \\
&= \sigma^4 - \frac{n\sigma^2}{2}\left(\frac{4\sigma^2}{n}\right) + \frac{n^2}{16}\left[\frac{16\sigma^4}{n^2}\left(1 + \frac{8}{5(n+1)}\right)\right], \\
&= \frac{8\sigma^4}{5(n+1)}.
\end{aligned}$$

From this fact and (9), we get

$$\begin{aligned}
|\triangle_1| &\le E|f'(D_n(\pi))|\left|\sigma^2 - \frac{n}{4}E^\pi\big(D_n(\pi') - D_n(\pi)\big)^2\right| \\
&\le \left(\frac{2}{\sigma^2}\right)\sqrt{E\left(\sigma^2 - \frac{n}{4}E^\pi\big(D_n(\pi') - D_n(\pi)\big)^2\right)^2} \\
&= \left(\frac{2}{\sigma^2}\right)\sqrt{\frac{8\sigma^4}{5(n+1)}} \\
&\le \frac{2.53}{\sqrt{n}}.
\end{aligned} \tag{24}$$

Bounding (21): By utilizing (1), (9), Lemma 2.3(1) and Lemma 2.3(3), we get

$$\begin{aligned}
|\triangle_2| &\le \frac{n}{4}\sqrt{\frac{\pi}{2}}\frac{1}{\sigma^3}\left|E\big(D_n(\pi') - D_n(\pi)\big)\int_0^{D_n(\pi')-D_n(\pi)} tdt\right| \\
&= \frac{n}{8}\sqrt{\frac{\pi}{2}}\frac{1}{\sigma^3}E|\big(D_n(\pi') - D_n(\pi)\big)^3| \\
&\le \frac{n}{8}\sqrt{\frac{\pi}{2}}\frac{1}{\sigma^3}E|\big(D_n(\pi') - D_n(\pi)\big)^2| \\
&\le \frac{n}{8}\sqrt{\frac{\pi}{2}}\frac{1}{\sigma^3}\left(\frac{4\sigma^2}{n}\right) \\
&\le \frac{1}{2}\sqrt{\frac{\pi}{2}}\frac{\sqrt{12}}{\sqrt{n}} \\
&\le \frac{2.18}{\sqrt{n}}.
\end{aligned} \tag{25}$$

Bounding (22): By the mean value theorem, there exists c between $D_n(\pi)$ and $D_n(\pi)+t$ such that,

$$f\big(D_n(\pi)+t\big)-f(D_n(\pi))=tf'(c),$$

for all $t\in[0,|D_n(\pi')-D_n(\pi)|]$. From this fact and (9), we obtain

$$\begin{aligned}|\triangle_3| &\le \frac{n}{4\sigma^2}\left|E\big(D_n(\pi')-D_n(\pi)\big)\int_0^{D_n(\pi')-D_n(\pi)}\big(D_n(\pi)-\mu\big)||f'||tdt\right|\\ &\le \frac{n}{4\sigma^4}E\left|\big(D_n(\pi)-\mu\big)\big(D_n(\pi')-D_n(\pi)\big)^3\right|,\end{aligned}$$

where $||f'||=\sup_{x\in\mathbb{R}}|f'(x)|$.

By Hölder's inequality, (1), (9), Lemma 2.3(1) and Lemma 2.3(3), we get

$$\begin{aligned}|\triangle_3| &\le \frac{n}{4\sigma^4}\sqrt{E\big(D_n(\pi)-\mu\big)^2}\sqrt{E\big(D_n(\pi')-D_n(\pi)\big)^6}\\ &\le \frac{n}{4\sigma^4}\sqrt{E\big(D_n(\pi)-\mu\big)^2}\sqrt{E\big(D_n(\pi')-D_n(\pi)\big)^2}\\ &= \frac{n}{4\sigma^4}\sqrt{\sigma^2}\sqrt{\frac{4\sigma^2}{n}}\\ &\le \frac{\sqrt{n}}{2}\frac{12}{n}\\ &\le \frac{6}{\sqrt{n}}.\end{aligned} \tag{26}$$

Bounding (23): Before we bound (23), we will show that,

$$\int_0^1\Big[h\big(D_n(\pi)+t\big)-h\big(D_n(\pi)\big)\Big]dt=\frac{1}{2}\Big[h\big(D_n(\pi)+1\big)-h\big(D_n(\pi)\big)\Big]. \tag{27}$$

Since $D_n(\pi)\in\{0,1,\dots,n-1\}$ and A is a non-empty subset of $\{0,1,\dots,n-1\}$, we can divide the proof into 4 cases.

Case 1: $D_n(\pi)\in A$ and $D_n(\pi)+1\in A$. Note that $h=h_{\bar{A}}$ where $\bar{A}=\bigcup_{z\in A}\Big[z-\frac{1}{2},z+\frac{1}{2}\Big)$. Then,

$$\int_0^1\Big[h(D_n(\pi)+t)-h(D_n(\pi))\Big]dt=\int_0^1(1-1)dt=0,$$

and

$$\frac{1}{2}\Big[h(D_n(\pi)+1)-h(D_n(\pi))\Big]=\frac{1}{2}(1-1)=0.$$

Case 2: $D_n(\pi)\in A$ and $D_n(\pi)+1\notin A$. Then,

$$\begin{aligned}&\int_0^1\Big[h(D_n(\pi)+t)-h(D_n(\pi))\Big]dt\\ &=\int_{[0,\frac{1}{2})}\Big[h(D_n(\pi)+t)-h(D_n(\pi))\Big]dt+\int_{[\frac{1}{2},1)}\Big[h(D_n(\pi)+t)-h(D_n(\pi))\Big]dt,\\ &=\int_{[0,\frac{1}{2})}(1-1)\,dt+\int_{[\frac{1}{2},1)}(0-1)\,dt,\\ &=-\frac{1}{2},\end{aligned}$$

and

$$\frac{1}{2}\Big[h(D_n(\pi)+1)-h(D_n(\pi))\Big] = \frac{1}{2}\,(0-1) = -\frac{1}{2}.$$

Case 3: $D_n(\pi)\notin A$ and $D_n(\pi)+1\in A$. Then,

$$\begin{aligned}
&\int_0^1 \Big[h(D_n(\pi)+t)-h(D_n(\pi))\Big]dt \\
&\quad= \int_{[0,\frac{1}{2})} \Big[h(D_n(\pi)+t)-h(D_n(\pi))\Big]dt + \int_{[\frac{1}{2},1)} \Big[h(D_n(\pi)+t)-h(D_n(\pi))\Big]dt, \\
&\quad= \int_{[0,\frac{1}{2})} (0-0)\,dt + \int_{[\frac{1}{2},1)} (1-0)\,dt, \\
&\quad= \frac{1}{2},
\end{aligned}$$

and

$$\frac{1}{2}\Big[h(D_n(\pi)+1)-h(D_n(\pi))\Big] = \frac{1}{2}\,(1-0) = \frac{1}{2}.$$

Case 4: $D_n(\pi)\notin A$ and $D_n(\pi)+1\notin A$. Then,

$$\int_0^1 \Big[h(D_n(\pi)+t)-h(D_n(\pi))\Big]dt = 0,$$

and

$$\frac{1}{2}\Big[h(D_n(\pi)+1)-h(D_n(\pi))\Big] = 0.$$

In a similar way that was used to derive (27), we can show that,

$$\int_{-1}^0 \Big[h(D_n(\pi)+t)-h(D_n(\pi))\Big]dt = \frac{1}{2}\Big[h(D_n(\pi)-1)-h(D_n(\pi))\Big]. \tag{28}$$

Since $D_n(\pi)$ and $D_n(\pi')$ are in $\{0,1,\ldots,n-1\}$ and $|D_n(\pi')-D_n(\pi)|\le 1$, we have $D_n(\pi')-D_n(\pi)\in\{-1,0,1\}$. This implies that,

$$\begin{aligned}
&\int_0^{D_n(\pi')-D_n(\pi)} [h(D_n(\pi)+t)-h(D_n(\pi))]\,dt \\
&\quad= \int_0^1 \Big\{h(D_n(\pi)+t)-h(D_n(\pi))\Big\}dtI\left(D_n(\pi')-D_n(\pi)=1\right) \\
&\qquad- \int_{-1}^0 \Big\{h(D_n(\pi)+t)-h(D_n(\pi))\Big\}dtI\left(D_n(\pi')-D_n(\pi)=-1\right).
\end{aligned}$$

From this fact and (27)–(28), we have to obtain that,

$$\begin{aligned}
&E(D_n(\pi') - D_n(\pi)) \int_0^{D_n(\pi')-D_n(\pi)} \Big[h(D_n(\pi)+t) - h(D_n(\pi))\Big] dt \\
&= E\Bigg[\int_0^1 \Big\{h(D_n(\pi)+t) - h(D_n(\pi))\Big\} dt I\left(D_n(\pi') - D_n(\pi) = 1\right) \\
&\qquad + \int_{-1}^0 \Big\{h(D_n(\pi)+t) - h(D_n(\pi))\Big\} dt I\left(D_n(\pi') - D_n(\pi) = -1\right)\Bigg], \\
&= \frac{1}{2} E\Bigg[\Big\{h(D_n(\pi)+1) - h(D_n(\pi))\Big\} I\left(D_n(\pi') - D_n(\pi) = 1\right) \\
&\qquad + \Big\{h(D_n(\pi)-1) - h(D_n(\pi))\Big\} I\left(D_n(\pi') - D_n(\pi) = -1\right)\Bigg], \\
&= \frac{1}{2} E\Bigg[\Big\{h(D_n(\pi')) - h(D_n(\pi))\Big\} I\left(D_n(\pi') - D_n(\pi) = 1\right) \\
&\qquad - \Big\{h(D_n(\pi)) - h(D_n(\pi'))\Big\} I\left(D_n(\pi) - D_n(\pi') = 1\right)\Bigg], \\
&= \frac{1}{2} E\Big[h(D_n(\pi')) - h(D_n(\pi))\Big], \\
&= \frac{1}{2} E\Big[g(D_n(\pi'), D_n(\pi))\Big], \\
&= 0,
\end{aligned}$$

where we use Proposition 2.1 with the antisymmetric function $g(x, y) = h(x) - h(y)$ in last equality. This implies that,

$$|\triangle_4| = 0. \tag{29}$$

By (11), (19)–(26) and (29), we conclude that,

$$\begin{aligned}
\left|P\left(D_n(\pi) \in A\right) - P(N^d_{\mu,\sigma^2} \in A)\right| &\leq \frac{2.53}{\sqrt{n}} + \frac{2.18}{\sqrt{n}} + \frac{6}{\sqrt{n}} \\
&= \frac{10.71}{\sqrt{n}}.
\end{aligned}$$

□

4 Conclusion

This study derived a bound for the difference between the number of descents and a discretized normal distribution under the total variation distance by utilizing Stein's method. The constant in our bound is sharper when compared to the standard normal, making Theorem 1.1 more suitable for evaluating the accuracy of this approximation. In future research, we aim to extend these criteria to cases involving the number of inversions.

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